ON MAGIC GRAPHS AND THEIR GENERATION

1. MAGIC GRAPHS

We shall consider a non-orientable finite graph $\mathbf{G} = [V(\mathbf{G}), E(\mathbf{G})]$ without loops, multiple edges or isolated vertices. If there exists a mapping $\mathcal{F}$ from the set of edges $E(\mathbf{G})$ into the set of positive real numbers such that

(i) distinct edges have distinct positive labels,
(ii) the sum of labels of the edges incident to a particular vertex is the same for all vertices

then the graph $\mathbf{G}$ is called *magic* and the mapping $\mathcal{F}$ is called a *magic labelling.* The analogous problem was solved for a labelling with non-negative real numbers.

A graph satisfies only the condition (i) is in [JTr] called *semimagic graph.* Adding multi-edges to a semimagic graph we can obtain regular graph, so in C.Berge's paper [Ber] it was called *regularizable graph.* If a graph $\mathbf{G}$ has perfect matching we can say that it is the semimagic graph with non-negative labelling with numbers 0 and 1 such that sum in each vertex is 1.

The study magic graphs was initiated by J.Sedláček [Se1], who was inspired by the relation between the magic square of order n and the complete bipartite graph $\mathbf{K}_{n,n}$.

Some sufficient conditions for the existence of magic graphs are established in [Mü], [Do3], and some constructions of magic graphs are in [St1], [Tr4].

Theorem 1. *(Trenkler* [Tr3]*)*
A connected graph $\mathbf{G}$ *with* n *vertices and* q *edges exists if and only if* $n = 2$ *and* $q = 1$ *or* $5 \le n$ *and* $\frac{5n}{4} < q \le \binom{n}{2}$.

A characterization of regular magic graphs in terms of even circuits was given by M.Doob [Do1]. Two different characterizations of magic graphs were published by R.H.Jeurissen [Je2] and S.Jezný and M.Trenkler [JTr].

First we shall formulate several necessary definitions.

A spanning subgraph $\mathbf{F}$ of the graph $\mathbf{G}$ is called a (1-2)-*factor* of $\mathbf{G}$ if each of its components is an isolated edge or a circuit. We say that a (1-2)-factor *separates* the edges e_1 and e_2, if one of them belongs to $\mathbf{F}$ and neither edge part nor circuit part contains both of them.

Theorem 2. *(Jezný, Trenkler* [JTr]*)*
A graph $\mathbf{G}$ *is magic if and only if,*

(i) *every edge belongs to a (1-2)-factor, and*
(ii) *every pair of edges* e_1, e_2 *is separated by a (1-2)-factor.*

Consequence.
A bipartite graph $\mathbf{G}$ *is magic if and only if for every pair of edges* e_1, e_2 *exists a 1-factor* $\mathbf{F}$ *such that* e_1 *belongs to* $\mathbf{F}$ *and* e_2 *does not belong to* $\mathbf{F}$.

Consequence. *(*[JTr]*)*
If $\mathbf{G}$ *is magic then there exists a magic labelling of* $\mathbf{G}$ *with positive integers.*

We shall denote by $\mathbf{P}_1\mathbf{P}_2$ a connected bipartite graph with the vertex-set equal to the union of independent sets P_1 and P_2. If P_1 and P_2 have the same number of vertices the graph is called *balanced.* An extra edge inserted between two vertices of the same part is called a *handle.* A *cross-bridge* is a pair of edges connecting disjoint graphs $\mathbf{P}_1\mathbf{P}_2$ and $\mathbf{Q}_1\mathbf{Q}_2$, the first of them from P_1 to Q_2, the second from P_2 to Q_1. As usual, $\Gamma(S)$ denotes the set of vertices adjacent to a vertex in set S.

Theorem 3. *(Jeurissen* [Je2]*)*
A connected bipartite graph $\mathbf{P}_1\mathbf{P}_2$ *is magic if and only if,*

(i) *it is balanced,*
(ii) $|\Gamma(S)| > |S|$ *for all* $S \neq P_1, P_2, \emptyset$, *and*
(iii) *does not consist of two balanced bipartite graphs connected by a cross-bridge.*

Theorem 4. *(Jeurissen* [Je2]*)*
A connected non-bipartite graph $\mathbf{G}$ *with vertex-set* V *is magic if and only if,*

(i) $|\Gamma(S)| > |S|$ *for all independent* $S \neq \emptyset$
(ii) $\mathbf{G}$ *is not balanced graph* $\mathbf{P}_1\mathbf{P}_2$ *provided with a handle at* P_1 *and one at* P_2, *and*
(iii) $\mathbf{G}$ *is not a graph consisting of a balanced subgraph* $\mathbf{P}_1\mathbf{P}_2$ *connected by one edge from* P_1 *and one from* P_2 *to another subgraph.*

A graph $\mathbf{G}^m$, arising from the graph $\mathbf{G}$ by joining every pair of vertices of $\mathbf{G}$ with the distance $d \leq m$ by a new edge, is called the *m-th power* of $\mathbf{G}$. Let $\mathbf{G}$ be a graph having 1-factor. If every edge of this 1-factor has exactly one vertex of degree 1, then $\mathbf{G}$ is called a I-*graph.* Using the previous Theorems 1 and 4 we can prove:

Theorem 5. *(Trenkler, Vetchý* [TrV]*)*
Let a graph $\mathbf{G}$ *has the order* $n \geq 5$. *The graph* $\mathbf{G}^2$ *is magic if and only* $\mathbf{G}$ *is not a I-graph and it is different from the path* $\mathbf{P}_5$ *of length 5. The graph* $\mathbf{G}^m$ *is magic for all* $m \geq 3$.

From the Theorem 5 it follows.

Theorem 6. *(*[TrV]*)*
A graph $\mathbf{G}^2$ *has a 1-factor or a factor consisting of isolated edges and exactly one triangle with a vertex in arbitrary vertex of* $\mathbf{G}$.

A special class of magic graphs is *prime-magic graphs.* Its edge are labeled by prime numbers. In [BH1], [St1], [Se2] are some results about prime-magic graphs $\mathbf{K}_{n,n}$.

2. SUPERMAGIC GRAPHS

If the edges of a magic graph $\mathbf{G}$ are labelled by consecutive integers of the interval $< 1, |V(\mathbf{G})| >$, then $\mathbf{G}$ is called *supermagic.*

From the fact that for every $n \neq 2$ there exists a magic square of order n it follows

Theorem 7.
A complete bipartite graph $\mathbf{K}_{n,n}$ *is supermagic if and only if* $2 \neq n$.

Some special classes of supermagic graphs are depicted in [Ba3], [BHL], [Se2] and [St2].

The following sufficient conditions was published or its are in preprints.

Theorem 8. *(Stewart* [St2]*)*
If n *is odd and* $5 < n \neq 0(mod4)$, *then* $\mathbf{K}_n$ *is supermagic.*

Stewart [St2] showed that a complete graph $\mathbf{K}_n$ is not supermagic when $n \equiv 0(mod4)$.

By *n-dimensional cube* $\mathbf{Q}_n$ we mean (n-1)-times cartesian products of the complete graph $\mathbf{K}_2$. The symbol $\mathbf{C}_n$ denotes the circle of length n.

J.Ivančo proved two next Theorems (private communication):

Theorem 9. *(Ivančo)*
The cube $\mathbf{Q}_n$ *is supermagic if and only if* $n = 1$ *or* $4 \leq n \equiv 0(mod2)$.

Theorem 10. *(Ivančo)*
If $n \geq 3$, *then the cartesian product of two copies of* $\mathbf{C}_n$ *is supermagic*

3. MAGIC DIGRAPHS

Let $\mathbf{D}$ ia a digraph without multiple edges and loops. We say that it is magic when the sum of labels of the arcs (oriented edges) initial and terminal to a particular vertex is the same for all vertices and distinct edges have distinct positive labels.

By a (1,1)-*factor* we mean a factor formed by vertex disjoin circuits. We say that *arc* a_1 *and* a_2 *are separated* by a (1,1)-factor $\mathbf{F}$ if either a_1 or a_2 belongs to $\mathbf{F}$.

Theorem 11. *(Borowiecki, Quintas* [BQ]*)*
A digraph $\mathbf{D}$ *is magic iff*

(i) *every arc of* $\mathbf{D}$ *belongs to some (1,1)-factor,*
(ii) *every couple of arc is separated by some (1,1)-factor.*

4. MAGIC HYPERGRAPHS

By a *complete k-uniform n-partite hypergraph* $\mathbf{H}_{k.n}$ we mean a hypergraph with $k.n$ vertices divided into k independent sets with the same number of vertices and n^k edges having exactly k vertices. The hypergraph $\mathbf{H}_{k.n}$ is *super-magic* if the edges are labeled by consecutive integers of the interval $< 1, n^k >$ such that the sum of labels of the edges incident to a particular vertex is the same for all vertices. The existence of such hypergraphs corresponds to the existence of k-dimensional magic hypercube of order $n \geq 2$, The following theorem was proved using orthogonal latin squares of order n.

Theorem 12. *(Trenkler)*
If $2 < n \neq 6$ *and* $2 < k$ *are integers, then the hypergraph* $\mathbf{H}_{k.n}$ *is supermagic.*

In [Tr2] is described the construction of magic cube of order $n \neq 2$ which corresponded with $\mathbf{H}_{...}$

Let $\mathbf{G}$ be a graph with n vertices denoted by $v_1 v_2, \ldots, v_n$ and let $\beta = (b_1, b_2, \ldots, b_n$ be an n-dimensional vector of positive real numbers. It is well known that if there exists a labelling of edges by positive real numbers such that the sum of labels incident with vertex v_i is b_i, then $\mathbf{G}$ has a *perfect β-matching.*

Theorem 13. *(Šándorová, Trenkler)*
Let $\mathbf{G}$ be a non-bipartite connected graphs with n vertices $v_1, v_2, \ldots, v_n$ and $\beta = (b_1, b_2, \ldots, b_n)$ be a vector of positive real numbers. The graph $\mathbf{G}$ is β-positive (β-non-negative) if and only if

$$\sum_{v_i \in S} b_i < (or \leq) \sum_{v_j \in \Gamma(S)} \quad \textit{for all independent} \quad S \neq \emptyset \quad \textit{of } \mathbf{G}$$

In ([ST1]) is proved a similar result for bipartite graphs. These results generalized Tutte's characterization of 2-matching of graphs.

Two characterizations of graph with perfect β-matching are given in [ST1] and [ST2]. A graph with perfect β-matching is called a *β-magic* if its edges are labeled by different positive numbers. The motivation for study of β-magic graphs is given in [Do2]. A characterization of β-magic graph is given in [ST2].

4. GENERALIZED-SUPERMAGIC GRAPHS

β-magic graph with consecutive labels of edges was studed in many papers. Its are different attractive names *(antimatic, graceful, harmonious)* (See for example [Gr], [BW], [Lih], etc.)

1. The concept of an antimagic graph introduce in Hartsfield and Ringel in [HR]

When the vertices of $\mathbf{G}$ can be indicated such that the coordinates of the vector β forms a arithmetical sequence with a minimal element a, a difference d and edges are labeled by integers from $< 1, 2, \ldots, |V(\mathbf{G})| >$ than the graph $\mathbf{G}$ is called *(a,d)-antimagic.* (See [BW1], [BW2], [BH2].)

2. From [BiR] it follows some results for the complete bipartite graphs $\mathbf{K}_{m,n}$, when we suppose that the sum of label in each vertex of the same independent set is the same.

5. REMARKS

1. In literature we can found another definition of magic graph. Some paper studed the labeling of edges and faces of planar graphs. For example see papers of Ko-Wei Lih [Lih] or Bača [Ba4] or many others authors.

2. Let A be a set. If we label the edges of $\mathbf{G}$ by different subsets of A such that the union of all subsets added to the edges incident to every vertex is A, then the graph is called *set-magic.* The problem of characterization of set-magic graphs was studied by some authors. For detail information see [Ach].

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